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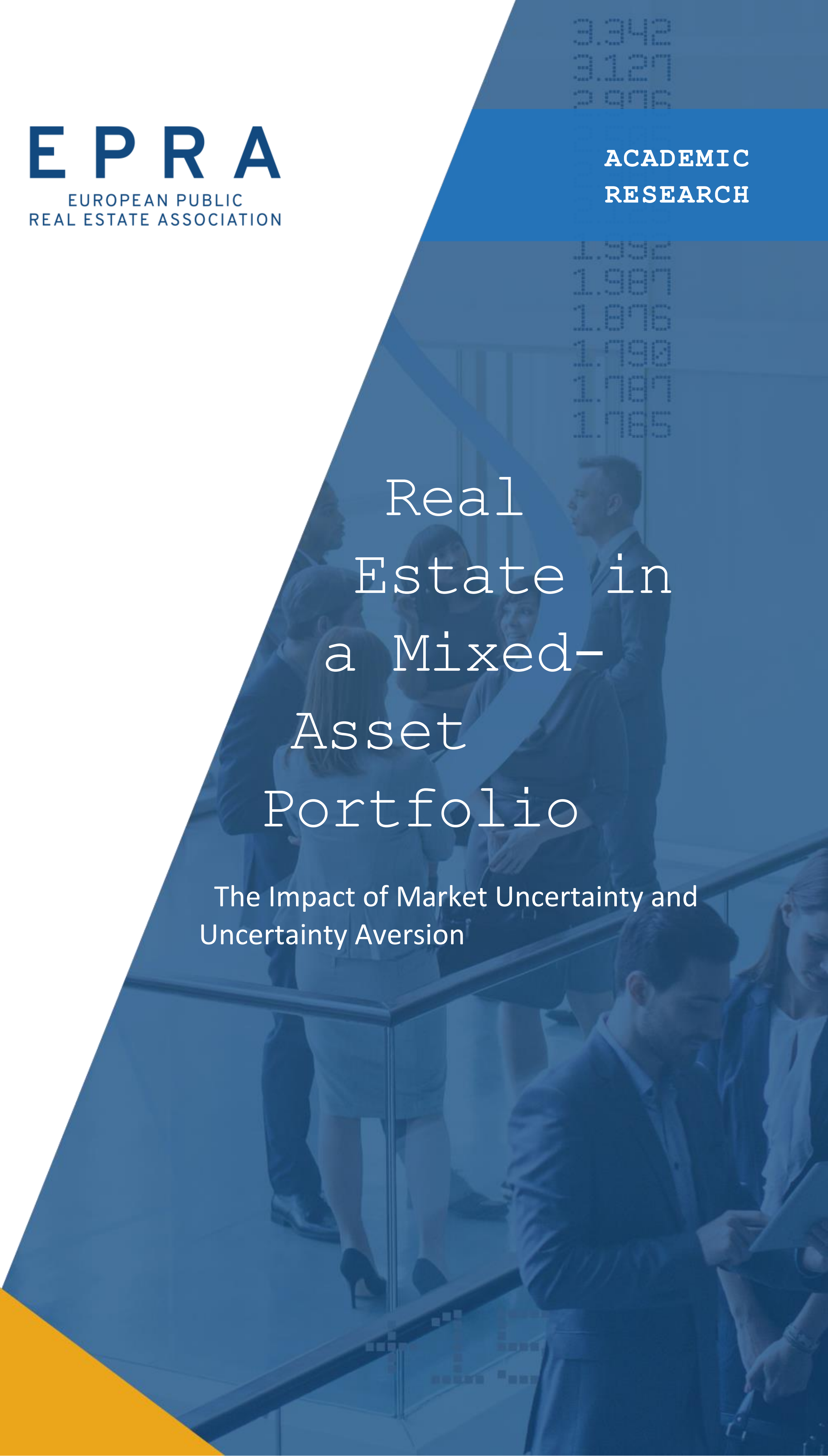


Real Estate in a Mixed-Asset Portfolio

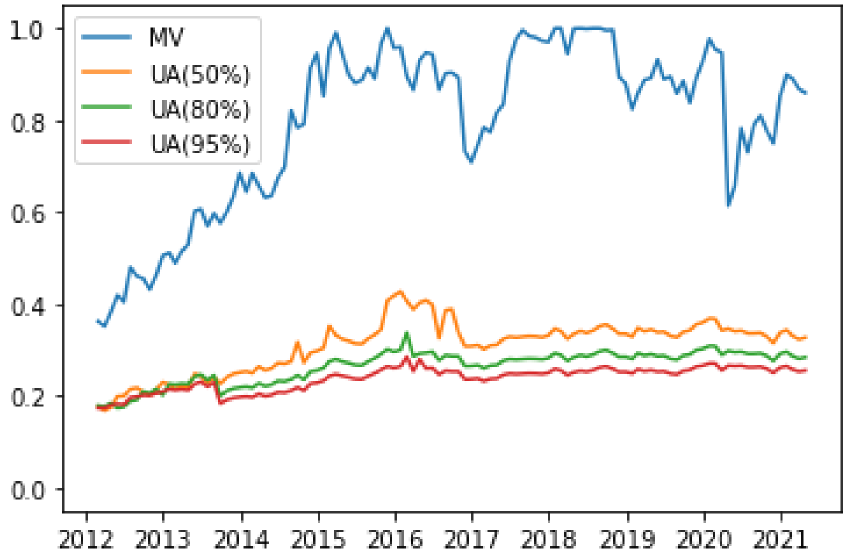
The Impact of Market Uncertainty and Uncertainty Aversion

January 2022

Executive Summary

The historical performance of European listed real estate and its historical correlation with traditional asset classes show how listed real estate may contribute to the performance and risk of a multi-asset portfolio. The classic mean-variance approach to determine the optimal allocation to listed real estate often yields extreme and unrealistic asset allocations and fluctuates significantly over time. Optimal allocations can be extremely volatile during a period of market uncertainty. By introducing the concept of 'uncertainty', this study adopts the uncertainty aversion approach to determine the optimal allocation to listed real estate. Optimal allocations to listed real estate show much greater stability under the uncertainty aversion approach compared to the mean-variance approach. This is illustrated in the diagram below:

Figure 1. Optimal allocations to listed real estate, anchored windows



Source: Author's calculation

Figure 1 shows the optimal allocations to European listed real estate over time using anchored windows (where the expected return, correlations and risk is based on the full history to that point since 2002). The blue line indicates the traditional "optimal" allocations to listed real estate under the classic mean-variance approach. The orange, green, and red lines indicate the optimal allocations to listed real estate under the uncertainty aversion approach with various levels of uncertainty. The red line has the highest aversion to uncertainty and shows the greatest stability in optimal allocations to listed real estate. An important implication of the more stable asset allocations over time is the major reduction in the need for portfolio rebalancing with an implied reduction in transaction costs. In addition, we show that the return-risk ratios are higher under the uncertainty aversion approach when compared to the mean-variance approach.

Given the magnitude of fluctuations in returns of the listed real estate sector in Europe, the estimated returns and volatilities can be very different across different periods. In other words, those point estimates are associated with uncertainty. By considering uncertainty of estimation using historical information and being averse to uncertainty, we show that, even with excellent past performance in listed real estate, it should be cautious to allocate too much to the sector. A transition from a listed real estate boom period to a decline period would significantly reduce the allocation to listed real estate if uncertainty aversion is not considered. Thus, the resultant uncertainty aversion portfolios are more stable over time and deliver a higher return/risk ratio. Given the heterogeneity in the performance of listed real estate among different European countries over different periods, an investor could form a portfolio including listed real estate from many countries. Since the uncertainty aversion approach gives us relatively stable allocations over time (even with many assets), the model will give us more modest shifts in allocations in listed real estate from one country to another over time. For example, the results of the multiple country analysis (UK, France and Germany) show a modest shift in listed real estate allocation from France to Germany after 2016. Thus, the uncertainty aversion model allows dynamic allocation of funds into the listed real estate sector with better performance without being too extreme in allocations and avoiding significant rebalancing costs.

Content table

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1. INTRODUCTION

Investors care about the expected returns of assets and the variance and covariances of asset returns. Those parameters are the key elements of portfolio selection models. In practical application of the classic mean-variance approach to portfolio selections, the risk and covariance parameters are typically estimated using historical data. In other words, it is assumed that the distribution of asset return is certain and can be estimated by using historical information. However, in practice, investors do not know the exact distribution of asset returns. Therefore, estimation of the return distribution using historical data contains estimation errors. It is more realistic to assume investors form prior beliefs about all possible distributions of asset returns and are averse to the uncertainty inherent in the future distribution. The aversion to uncertainty described here is called ambiguity aversion in some studies (Chen and Epstein, 2002; Epstein and Miao, 2003; Garlappi et al., 2007). Throughout this paper, the terms “uncertainty” and “ambiguity” are treated as equivalent.

It is important to distinguish between the “risk” element and the “uncertainty” element of asset returns. The risk element refers to the possible fluctuations of asset returns given that the distribution of those returns is known. The uncertainty element refers to the uncertainty about the distribution of asset returns. To illustrate the difference between risk and uncertainty, a good example would be rolling a die. A game of rolling a fair die is only associated with risk. Since the distribution of outcomes of rolling a fair die is known, there is no uncertainty about the distribution and the risk refers to the fluctuation of outcomes for that known distribution. However, for asset returns, investors have to consider both risk and uncertainty because the distribution of asset returns is unknown and there is risk associated with any given distribution of asset returns.

One criticism of the classic mean-variance approach is that the estimates of the distribution of asset returns are sensitive to the choice of historical data. Thus, even with the same performance expectations, the optimal allocation of assets fluctuates over time depending on the choice of historical period to estimate risk (and co-variance). In addition, extreme optimal allocations (“corner solutions”) often arise in the classic mean-variance approach. Thus, by considering estimation error and aversion to uncertainty, this study examines how the portfolio allocations in the listed real estate sector differ under the uncertainty aversion framework compared to the classic mean-variance framework. In addition, this study analyses how the differences in portfolio allocations under the two frameworks vary over time (using different historical periods – allowing risk, returns and covariance to change).

The report is organised as follows:

- **Section 2** reviews the background to uncertainty aversion
- **Section 3** sets out the data and methods used in this study
- **Section 4** examines the optimal allocations of European listed real estate
- **Section 5** conducts robustness checks by examining the optimal allocations to listed real estate in individual countries
- **Section 6** presents our conclusions

2. BACKGROUND TO UNCERTAINTY AVERSION

Daniel Ellsberg’s (1961) experiment demonstrated that risk and uncertainty are fundamentally different. Broadly speaking, risk refers to the case where the probability distribution over the state of the world is known, while uncertainty refers to the case where there is imperfect knowledge about the probability distribution. In other words, the situation of uncertainty cannot be expressed using a single probability distribution and is thus characterised by a set of probability distributions. One of Daniel Ellsberg’s experiments is the following: there are two urns, each containing 100 balls of either red or black colour. Urn A contains 50 black balls and 50 red balls, while there is no information about the proportion of red and black for urn B. The experiment results showed that people betting on a colour often prefer to draw a ball from urn A rather than urn B. This is a phenomenon that people are trying to avoid uncertainty and are, thus, uncertainty-averse. Surveys among investors (Olsen and Troughton, 2000) and laboratory experiments (Sarin and Weber, 1993) showed the presence of uncertainty aversion and uncertainty plays a role in the financial decision-making processes.

This concept of uncertainty has been widely acknowledged in finance and portfolio theories. In most of our real-life situations, asset returns are uncertain (Uppal and Wang, 2003). Thus, there are various theoretical models to explore the implication of uncertainty on portfolio allocations. For example, Epstein and Miao (2003) presented a model of international portfolio choices in a setting of two agents who differ in their ambiguity about returns. Maenhout (2004) studied portfolio choice between a riskless and a single risky asset under the situation of uncertainty. Uppal and Wang (2003) developed a model that allows for differences in the degree of ambiguity across different asset returns and explored the impact of uncertainty for portfolio diversification.

One useful feature of those theoretical models is a way of dealing with estimation errors in expected returns in the portfolio selection process. However, how those theoretical models can be used in practice is still problematic. Garlappi et al. (2007) provided a solution to applying the standard theoretical model of portfolio selection to practical problems under the situation of uncertainty. Based on eight international equity indices, they showed that, compared with portfolios from the classic mean-variance approach, uncertainty aversion portfolios are more stable over time and deliver a higher out-of-sample return-risk ratio.

3. DATA AND METHODOLOGY

For the main analysis in this study, four asset classes are used: European listed real estate (FTSE EPRA/Nareit Developed Europe Index), European stocks (S&P European Index), European government bonds (J.P. Morgan GBI Development Market Europe Index), and commodities (Bloomberg Commodity Index). The analysis is based upon the monthly total returns of those four assets from February 2002 to April 2021. As a robustness check, we focus on assets from France, Germany and the UK. Details are explained in the robustness check section.

The theoretical frameworks of the classic mean-variance approach and the uncertainty aversion approach are presented in Appendix A. Under the uncertainty aversion approach, the expected return is not a point estimate but an interval estimate. In particular, the expected return for each asset lies within a specific confidence interval around its estimated value. The confidence interval depends on the degree of the investor's uncertainty aversion and the uncertainty of the asset return[1]. The intuition is that if the confidence interval of the expected returns of a particular asset is large, this indicates that the estimated mean return is imprecisely estimated, then investors should rely less on the estimated mean return and reduce the allocation in this asset. For the optimisation problems in this study, we assume that there is no risk-free asset and short-selling is not allowed.

To analyse how the optimal allocations of assets vary over time under both the classic mean-variance approach and the uncertainty aversion approach, rolling windows and anchored windows are used in this study. In the anchored window approach, we start from a 120-month window starting at the earliest date in the data set, add one month each step, and use this historical information to estimate optimal asset allocations. We update the estimated optimal asset allocations each month for every additional month of information available. In the case of the anchored window, at any given point of time, the estimated optimal asset allocations are based on all the available historical information. In the rolling window approach, by contrast, the estimated optimal asset allocations are always based on the historical information of the preceding 120 months. The anchored window approach maximises the amount of data used in the allocation decision which should result in a lower aggregate measure of volatility as the sample size increases. However, it will be insensitive to any changes in the relationships between asset classes and the relative risk-return performance that emerge over time, and to any structural breaks, which will be better captured by the rolling window approach.

We assess the optimal allocations of various strategies by looking at their ex-post or out-of-sample performance. For both anchored window and rolling window, at any given point of time, we estimate the optimal portfolio weights for each strategy and use these portfolio weights to calculate the return of the portfolio in the next month. The resulting out-of-sample period spans from March 2012 to April 2021 (110 observations). Based on those 110 observations for each strategy, we calculate the average return, standard deviation and return-to-standard deviation (return/risk) ratio.

[1] The investors' uncertainty aversion is common across assets and uncertainty of the asset return is asset specific. This is similar to the concept on the difference between investor's risk aversion and risk for an asset class. Risk aversion is common across assets and the risk of asset return is asset specific.

4. BROAD EUROPEAN ASSET CLASSES ANALYSIS – OPTIMAL ALLOCATIONS TO LISTED REAL ESTATE

Table 1 reports the summary statistics for the total returns of European listed real estate, European stocks, European government bonds and commodities. Listed real estate delivers the highest return but is the riskiest asset over this period. As expected, government bonds have the lowest return (other than commodities) and are the least risky asset. Stocks lie between listed real estate and government bonds in terms of both return and risk characteristics. Commodities have a low return and a relatively high risk over this period. However, the low correlations between commodities and other asset classes suggest commodities might still provide diversification benefits. The correlation coefficient between listed real estate and the general stock market is 0.56 (on a monthly basis), thus indicating that there is a potential diversification benefit from including listed real estate in a portfolio alongside stocks. Government bonds are negatively correlated with all other asset classes.

Table 1. Summary statistics on performance of asset classes

	Listed Real Estate	Stocks	Government Bonds	Commodities
Monthly Return	0.74%	0.58%	0.27%	0.20%
Monthly σ	5.00%	5.61%	3.06%	4.49%
Annualised return	8.93%	6.94%	3.25%	2.38%
Annualised σ	17.3%	19.4%	10.6%	15.5%
Correlation between monthly returns				
Listed real estate	1.00			
Stocks	0.56	1.00		
Government bonds	-0.09	-0.10	1.00	-0.32
Commodities	0.28	0.37	-0.32	1.00

Note: σ indicates standard deviation.

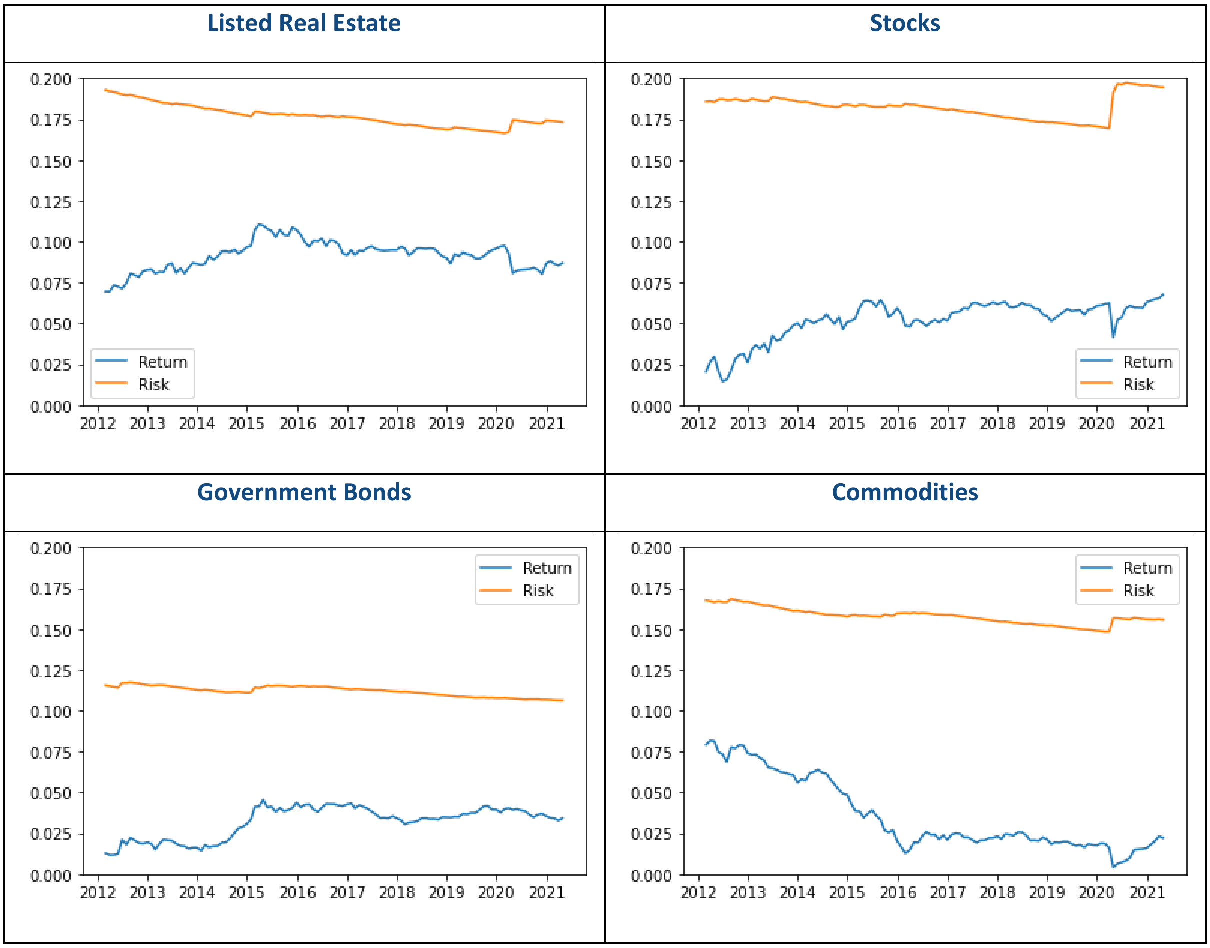
4.1. ANCHORED WINDOW ANALYSIS

The analysis in this subsection is based on anchored window analyses starting from an initial 120-month window (e.g. estimation of returns and risk starts from the earliest point in the dataset with a ten-year analysis period, then proceeds by adding an additional month each step). Historic annualised average returns and standard deviations are presented in Figure 1. Listed real estate returns and standard deviations were relatively stable over time using anchored windows. Correlations between listed real estate and other asset classes are presented in Figure 2. The estimated correlations were stable over time using anchored windows.

Under the classic mean-variance approach, for a given set of assets and assumptions, an efficient frontier (the best return for any given level of risk) can be estimated. If return expectations are based on historical returns, then investors with a higher risk tolerance will allocate more into assets with higher historical returns, in this case, listed real estate. To select a sensible risk aversion parameter, we choose the risk aversion parameter γ equal to 2[2]. For a given risk aversion parameter, we vary the uncertainty parameter ϵ to analyse the effect of level of uncertainty.

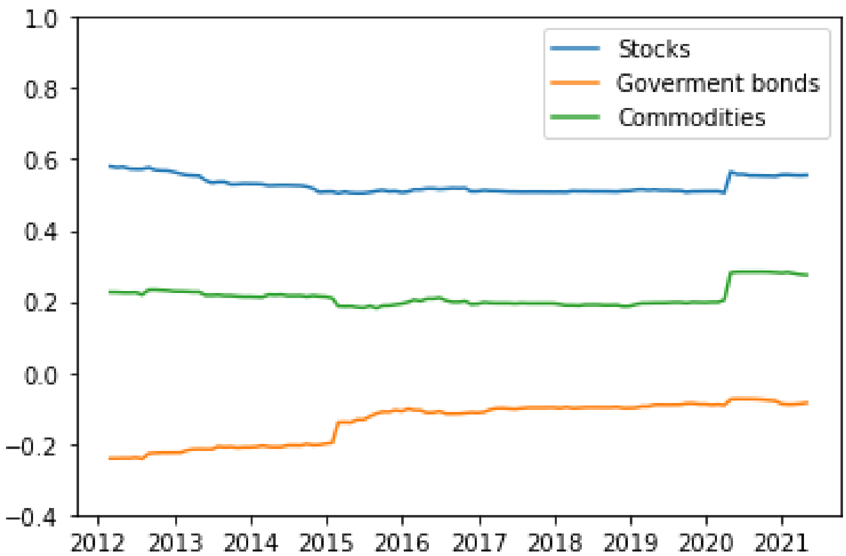
[2] γ is the risk aversion parameter that appear in both the classic mean-variance model and uncertainty aversion model in Appendix A. The larger the value of γ, the more risk averse the investor is. Each value of γ is associated with an optimal portfolio that lies on the efficient frontier. Under the classic mean-variance approach in this study, choosing $\gamma = 1$ would result a 100% allocation to listed real estate in most of the time periods. Thus, it would be more appropriate to choose $\gamma = 2$.

Figure 1. Anchored windows annualised average returns and standard deviations



Source: Author's calculation

Figure 2. Anchored windows correlations between listed real estate and other asset classes

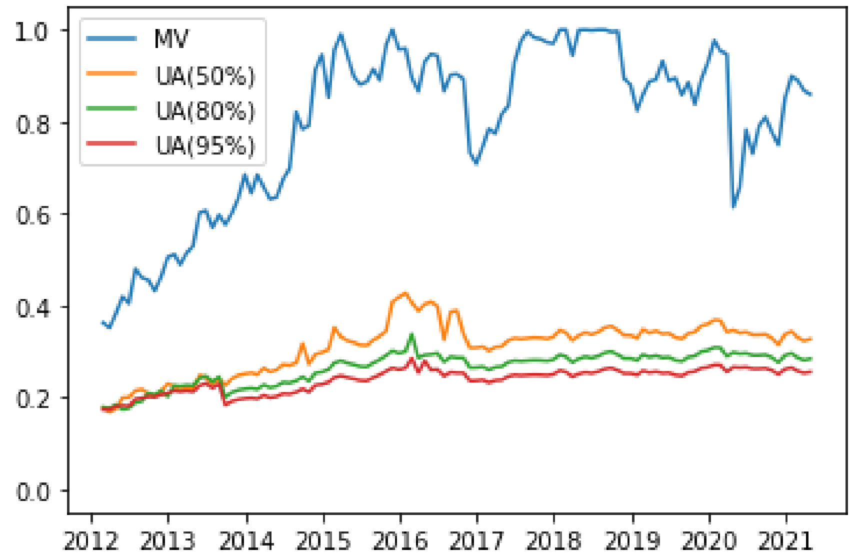


Source: Author's calculation

Uncertainty in this study is characterised by the confidence interval around the estimate of expected return. Thus, the choice of level of uncertainty is the same as specifying a confidence interval around the estimated expected return. An intuitive way to understand the confidence interval is the following: An investor can report their best estimates of expected returns. He/she can, at the same time, report the uncertainty of his/her estimates by stating that the confidence level is 95% that the estimated expected return lies within a specific interval around that best estimate. The uncertainty parameter ϵ can be interpreted as the size of a confidence interval. A larger ϵ means a higher confidence interval (a larger range of returns), thus a higher level of uncertainty (detailed explanations are provided in Appendix A). In this study, we choose the uncertainty parameter ϵ such that uncertainties about expected returns are given by the 50%, 80%, and 95% confidence intervals.

Figure 3 shows optimal allocations to listed real estate over time. The blue line indicates the optimal allocations to listed real estate under the classic mean-variance approach. The orange, green, and red lines indicate the optimal allocations to listed real estate under the uncertainty aversion approach with confidence intervals of 50%, 80%, and 95%, respectively.

Figure 3. Optimal allocations to listed real estate, anchored windows



Source: Author's calculation

Table 2 shows the summary statistics of optimal allocations to listed real estate in different scenarios. Under the classic mean-variance approach, using historical returns as expected returns, the average optimal allocation to listed real estate is an unrealistic 80%, far out-of-line with institutional allocations. The minimum and maximum optimal allocations to listed real estate over the period are 35% and 100%, respectively. Even though the return and risk of listed real estate and the correlation with other assets are fairly stable over time, the classic mean-variance approach produces a significant variation in the optimal allocations which would force frequent, significant and costly portfolio rebalancing. The standard deviation of the optimal allocation to listed real estate is 18%. However, under the uncertainty aversion approach, as the uncertainty (confidence interval) increases, the optimal allocations of listed real estate become more stable over time. There is a significant reduction in the standard deviation of optimal allocations to real estate with uncertainty aversion. For example, with a confidence interval of 95%, the average optimal allocation to listed real estate is 24%. The minimum and maximum optimal allocations to listed real estate are 17% and 29%, respectively and the standard deviation of optimal allocation to listed real estate is only 3%. The associated optimal allocations of stocks and government bonds are reported in Figure B1 in Appendix B.

Table 2. Summary statistics of optimal allocations to European listed real estate, anchored windows

	Mean	σ	Min	Max
Mean-Variance	80%	18%	35%	100%
Uncertainty Aversion (50% confidence interval)	31%	6%	17%	43%
Uncertainty Aversion (80% confidence interval)	26%	4%	17%	34%
Uncertainty Aversion (95% confidence interval)	24%	3%	17%	29%

Note: σ indicates standard deviation.

Table 3 reports the out-of-sample performance of our estimated optimal allocations. Compared to the portfolios constructed using the mean-variance approach, there are significant reductions in both return and volatility of the portfolios constructed using the uncertainty aversion approach. However, the return-risk ratios (measured by mean return-to-standard deviation ratio)[3] of the uncertainty aversion portfolios are higher indicating superior risk-adjusted return performance.

Table 3. Out-of-sample performance, anchored windows

	Mean	σ	$\frac{\text{Mean}}{\sigma}$
Mean-Variance	0.50%	3.90%	0.1294
Uncertainty Aversion (50% confidence interval)	0.36%	2.45%	0.1463
Uncertainty Aversion (80% confidence interval)	0.36%	2.38%	0.1497
Uncertainty Aversion (95% confidence interval)	0.35%	2.34%	0.1503

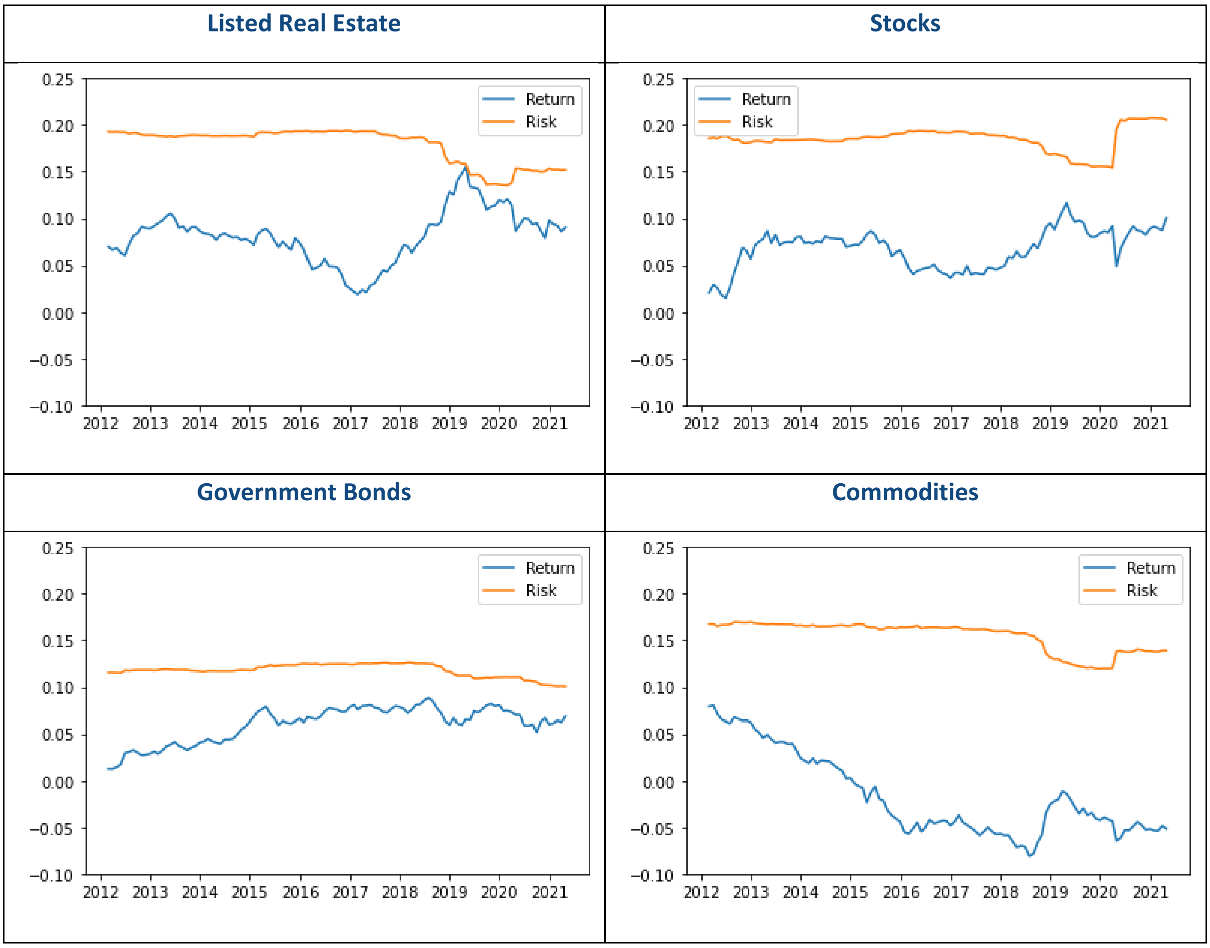
Note: σ indicates standard deviation.

4.2. ROLLING WINDOW ANALYSIS

The analysis in this subsection is based on 120-month rolling windows (that is, successive ten-year analysis periods are created by dropping the first month from the previous analysis and adding an additional month). Historic annualised average returns and standard deviations are presented in Figure 4. Focussing on the listed real estate, the 10-year average returns fell between 2013 and 2017 and increased between 2017 and 2019. Correlations between listed real estate and other asset classes are presented in Figure 5. The estimated correlations between listed real estate and other asset classes were stable before 2019 but there is more variation from 2019.

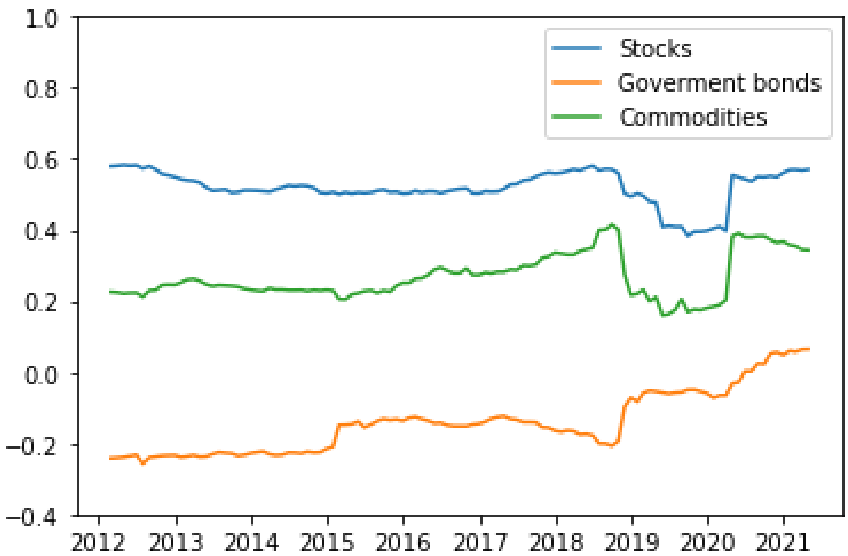
[3] We follow Garlappi et al. (2007) for the calculations of all the out-of-sample performance indicators.

Figure 4. 120-month rolling windows annualised average returns and standard deviations



Source: Author’s calculation

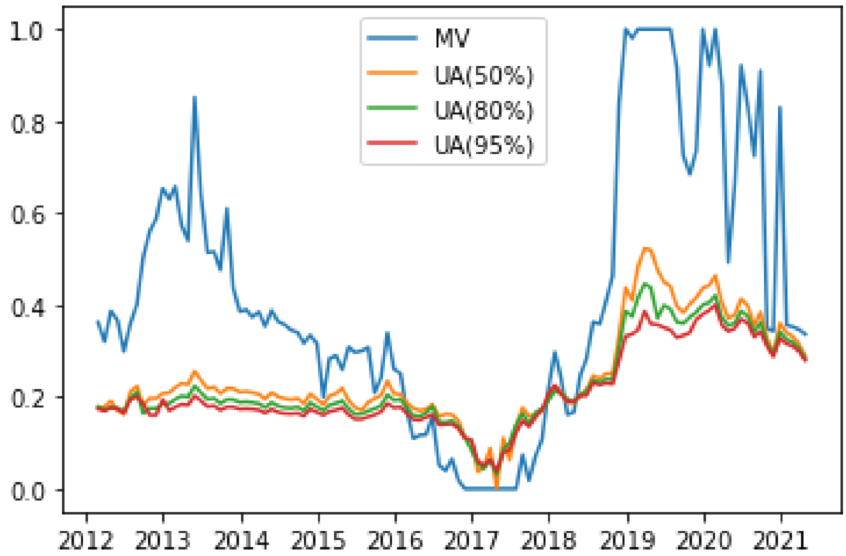
Figure 5. 120-month rolling windows correlation between listed real estate and other asset classes



Source: Author’s calculation

Figure 6 shows optimal allocations of listed real estate over time. The blue line indicates the optimal allocations of listed real estate under the classic mean-variance approach. The orange, green, and red lines indicate the optimal allocations of listed real estate under the uncertainty aversion approach with confidence intervals of 50%, 80%, and 95%, respectively.

Figure 6. Optimal allocations to listed real estate, rolling windows



Source: Author’s calculation

Table 4 shows the summary statistics of optimal allocations to listed real estate with the different approaches and inputs. Under the classic mean-variance approach, using historical returns as expected returns, the average optimal allocation to listed real estate is 42%. The minimum and maximum optimal allocation to listed real estate over the period are 0% and 100%, respectively. The standard deviation of optimal allocation to listed real estate is 30%. Not only are many of the allocations significantly out of line with investor behaviour and extreme, but the suggested allocations also require very significant and frequent costly portfolio rebalancing. As expected, the allocations are much more volatile than for the anchored windows.

Under the uncertainty aversion approach, as the uncertainty (confidence interval) increases, the optimal allocations of listed real estate are more stable over time. There is a significant reduction in the standard deviation of optimal allocations to real estate with uncertainty aversion. For example, with a confidence interval of 95%, the average optimal allocation to listed real estate is 21%. The minimum and maximum optimal allocations to listed real estate are 4% and 40%, respectively. The standard deviation of optimal allocation to listed real estate is just 9%. With rolling windows, the fluctuations of optimal allocations to listed real estate are larger compared to estimations in anchored windows, reflecting the less stable inputs to the optimisation model. The associated optimal allocations of stocks and government bonds are reported in Figure B2 in Appendix B.

Table 4. Summary statistics of optimal allocations to European listed real estate, rolling windows

	Mean	σ	Min	Max
Mean-Variance	42%	30%	0%	100%
Uncertainty Aversion (50% confidence interval)	24%	11%	0%	52%
Uncertainty Aversion (80% confidence interval)	22%	9%	3%	45%
Uncertainty Aversion (95% confidence interval)	21%	9%	4%	40%

Note: σ indicates standard deviation.

Table 5 reports the out-of-sample performance of our estimated optimal allocations. Compared to the portfolios constructed using the mean-variance approach, there is a very significant reduction in the volatility of the portfolios constructed using the uncertainty aversion approach. However, the average returns under the two frameworks are very similar. Thus, there is a significant increase in the return/risk ratio using the uncertainty aversion approach (this is before allowing for any additional rebalancing costs with the mean-variance approach).

Table 5. Out-of-sample performance, rolling windows

	Mean	σ	$\frac{\text{Mean}}{\sigma}$
Mean-Variance	0.44%	3.54%	0.1262
Uncertainty Aversion (50% confidence interval)	0.47%	2.62%	0.1813
Uncertainty Aversion (80% confidence interval)	0.45%	2.55%	0.1772
Uncertainty Aversion (95% confidence interval)	0.42%	2.50%	0.1693

Note: σ indicates standard deviation.

4.3. DISCUSSION

Optimal allocations to listed real estate show much greater stability under the uncertainty aversion approach when compared to the mean-variance approach. With rolling windows, the fluctuations of optimal allocations to listed real estate using the classic mean-variance approach are much larger compared with anchored windows. This is due to the greater variation in returns and less stable inputs for the optimisation. However, the difference in standard deviations of optimal allocations between rolling windows and anchored windows is much smaller under the uncertainty aversion approach. This reemphasises the fact that the uncertainty aversion approach can better deal with fluctuating inputs and produces more stable asset allocations.

Based on the out-of-sample performances in both rolling windows and anchored windows, the return/risk ratio increases by allowing for uncertainty aversion. The magnitude of the increase is much higher in the case of rolling windows. Although the result is consistent with Garlappi et al. (2007), this does not guarantee that the uncertainty aversion approach always yields a higher out-of-sample return/risk ratio.

Under the uncertainty aversion approach, comparing the out-of-sample performance between anchored windows and rolling windows, the return/risk ratios under rolling windows are higher. This brings us back to the issue of how much historical information one should use when constructing an optimal portfolio. The benefit of using an anchored window is that we can utilise all the historical information. However, it ignores structural changes and puts less weight on recent market movements when forming expectations. Thus, recent trends and investment opportunities might be missed. Investment based on rolling window information might be too aggressive, and in fact, the out-of-sample return/risk ratio of the mean-variance approach in rolling windows is slightly lower than that in anchored windows. Although under the uncertainty aversion framework, rolling windows produce a higher out-of-sample return/risk ratio than anchored windows we do not think it is safe to conclude that rolling windows are preferable. This result is largely a result of very recent performance and before the last few years rolling windows did not deliver superior performance.

An important implication of more stable asset allocations over time is less requirement for portfolio rebalancing. Since frequently rebalancing portfolio typically incurs transaction costs, the overall return is likely to be significantly lower for a portfolio with very unstable allocations. The mean-variance approach suggests heavy portfolio rebalancing, especially in the case of rolling windows. If transaction costs are considered, the out-of-sample return/risk ratio under the uncertainty aversion approach should be even better than that under the mean-variance approach.

Although the optimal allocation to listed real estate under the uncertainty aversion approach is still higher than the actual allocation of institutional investors, the allocation is more sensible than with the mean-variance approach. Part of the reason that the optimal allocation to listed real estate is high in this study is due to the strong performance of European listed real estate as shown in Figures 1 and 4. Since the purpose of this study is to show the stability of asset allocations under the uncertainty aversion approach, whether the asset allocation number is sensible will not be discussed in detail.

5. ROBUSTNESS CHECK – INDIVIDUAL COUNTRY ANALYSIS

Analysis in the previous section is based on the broad European asset classes, in this section, we perform a similar analysis for three individual countries: France, Germany and UK. We choose these countries for several reasons. First, those countries are three of the largest economies in Europe. Second, the historic performance of the listed real estate sector in those three countries has been different. The French listed real estate sector has delivered relatively strong risk-adjusted performance over the past three decades, whereas the German and the UK real estate sector delivered relatively poor risk-adjusted performance. Thus, this allows us to understand better how the uncertainty aversion approach works under different scenarios. Third, there is a longer series of data for those three countries: we have ten years more data than the analysis in the previous section. In addition, given the longer series of data, it allows us to test the effect of changing frequency. We use information from quarterly data and analysis the effect on our results.

Like the choice of asset classes for the broad European market, we use four asset classes for the individual country analysis. For France, the asset classes considered are listed real estate (FTSE EPRA/Nareit France Index), stocks (MSCI France Index), government bonds (ICE BofA France Government Index), and commodities (Bloomberg Commodity Index) are used. For Germany, we use listed real estate (FTSE EPRA/Nareit Germany Index), stocks (MSCI Germany Index), government bonds (ICE BofA Germany Government Index), and commodities (Bloomberg Commodity Index) are used. For the UK, we use list real estate (FTSE EPRA/Nareit UK index), stocks (MSCI UK index), government bonds (ICE BofA UK Gilt Index), and commodities (Bloomberg Commodity Index). The analysis is based upon the monthly total returns of those assets from February 1991 to April 2021.

Table 6 reports the summary statistics of total returns of assets in France. The pattern is similar to the broad European market, government bonds have the lowest return (except commodities) and are the least risky asset. Stocks lie between listed real estate and Government bonds in terms of both return and risk characteristics. The correlation coefficient between listed real estate and the general stock market is 0.59.

Table 6. Summary statistics on performance of asset classes in France

	Listed Real Estate	**Stocks**	**Government Bonds**	**Commodities**
Monthly Return	0.95%	0.81%	0.49%	0.27%
Monthly σ	5.47%	5.16%	1.21%	4.01%
Annualised return	11.4%	9.73%	5.85%	3.24%
Annualised σ	18.9%	17.9%	4.20%	13.9%
Correlation between monthly returns				
Listed real estate	1.00			
Stocks	0.59	1.00		
Government bonds	0.13	-0.02	1.00	
Commodities	0.19	0.13	-0.09	1.00

Note: σ indicates standard deviation.

Table 7 reports the summary statistics of total returns of assets in Germany. The pattern is quite different from the broad European market; compared to stocks, listed real estate has lower return but higher risk. Compared to France's listed real estate, Germany's listed real estate has provided a much lower return and much higher risk. This indicates the relatively poor performance of Germany's listed real estate in the past three decades. The correlation coefficient between listed real estate and the general stock market is 0.42 which is lower than found with the broad Europe and France indices

Table 7. Summary statistics on performance of asset classes in Germany

	Listed Real Estate	Stocks	Government Bonds	Commodities
Monthly Return	0.71%	0.81%	0.44%	0.27%
Monthly σ	6.64%	5.78%	1.07%	4.01%
Annualised return	8.49%	9.75%	5.26%	3.24%
Annualised σ	23.0%	20.0%	3.71%	13.9%
Correlation between monthly returns				
Listed real estate	1.00			
Stocks	0.42	1.00		
Government bonds	-0.07	-0.21	1.00	
Commodities	0.07	0.13	-0.15	1.00

Note: σ indicates standard deviation.

Table 8 reports the summary statistics of total returns of assets in the UK. The pattern is similar to Germany; compared to stocks, listed real estate has lower return but higher risk. Compared to Germany's listed real estate, UK listed real estate has provided a lower return and lower risk. Compared to France's listed real estate, UK listed real estate has provided a much lower return but similar risk. The correlation coefficient between listed real estate and the general stock market is 0.61.

Table 8. Summary statistics on performance of asset classes in the UK

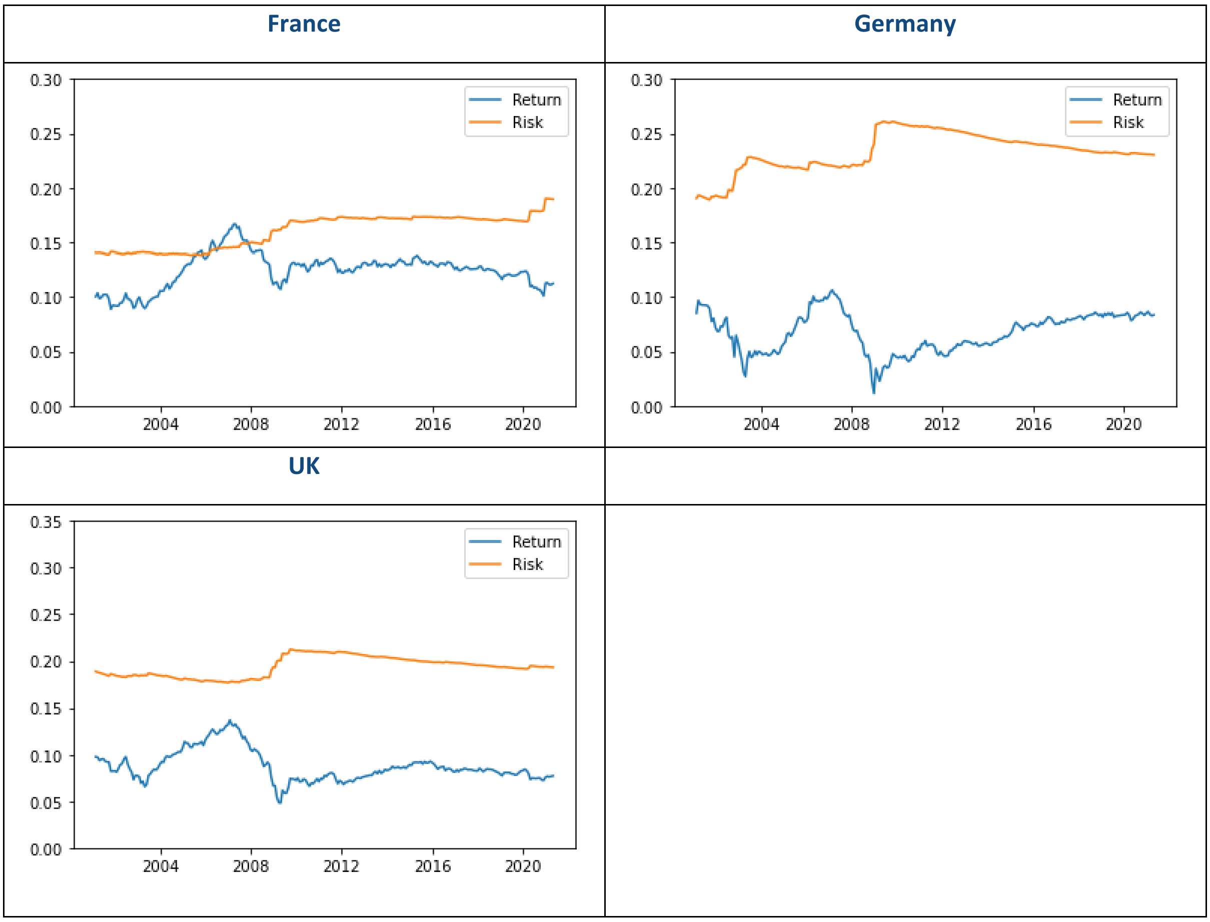
	Listed Real Estate	Stocks	Government Bonds	Commodities
Monthly Return	0.66%	0.68%	0.58%	0.27%
Monthly σ	5.57%	4.00%	1.71%	4.01%
Annualised return	7.89%	8.17%	7.00%	3.24%
Annualised σ	19.30%	13.87%	5.94%	13.91%
Correlation between monthly returns				
Listed real estate	1.00			
Stocks	0.61	1.00		
Government bonds	0.13	0.06	1.00	
Commodities	0.21	0.16	-0.17	1.00

Note: σ indicates standard deviation.

5.1. ANCHORED WINDOW ANALYSIS

Figure 7 shows the historical annualised average returns and standard deviations of listed real estate returns using anchored windows. For all subperiods, France's listed real estate had higher average returns and lower risks compared with Germany's listed real estate. The returns patterns are similar between France and UK, but the risks of UK listed real estate are much higher than the France's listed real estate across all subperiods. There were sharp increases in average returns of France's and UK listed real estates during 2004-2007. There were two sharp declines in average returns of Germany's listed real estate, one before 2004 and one before 2009.

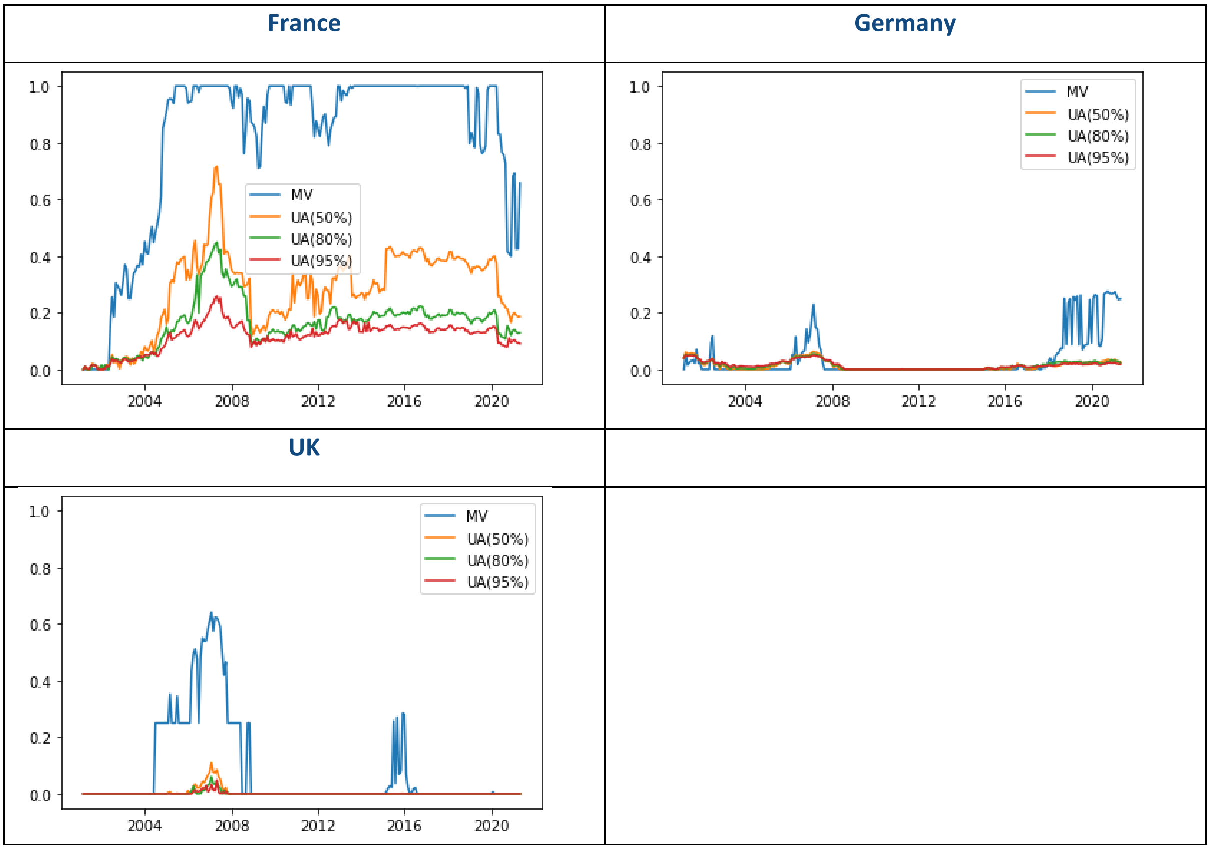
Figure 7. Anchored windows annualised average returns and standard deviations of listed real estate



Source: Author's calculation

Figure 8 shows the optimal allocations to listed real estate using anchored windows with the summary statistics reported in Table 9. Focussing on France, the average optimal allocations to listed real estate and standard deviations of allocations significantly drop when allowing for uncertainty aversion. As the uncertainty (confidence interval) increases, the optimal allocations of listed real estate are more stable over time. There is a peak in the allocation to listed real estate in 2007, due to the superior historical average returns at that time. Focussing on Germany, there are barely any allocations to listed real estate, due to the persistently poor risk-adjusted performance of the listed real estate sector in Germany. There are two phases of recovery of the market. During 2006-2007, the average historical return is around 10%. After 2017, the average historical return is above 8%. As a result, there are some positive allocations to listed real estate during 2006-2007 and after 2017. Part of the reason for the very low allocation to the listed real estate is the use of anchored windows. Since we are utilising all the historical information, even when the market started to recover, the low returns in the past still drive down the average returns. Focussing on the UK, similar to the case in Germany, there are barely any allocations to listed real estate due to the persistently poor risk-adjusted performance of the listed real estate sector. There are some positive allocations to listed real estate during 2006-2007, this is due to the superior performance of UK listed real estate before 2007.

Figure 8. Optimal allocations to listed real estate, anchored windows



Source: Author's calculation

Table 9. Summary statistics of optimal allocations to European listed real estate, anchored windows

	Mean	σ	Min	Max
Panel A: France				
Mean-Variance	76%	31%	0%	100%
Uncertainty Aversion (50% confidence interval)	27%	14%	0%	65%
Uncertainty Aversion (80% confidence interval)	17%	9%	0%	44%
Uncertainty Aversion (95% confidence interval)	12%	6%	0%	27%
Panel B: Germany				
Mean-Variance	4%	8%	0%	28%
Uncertainty Aversion (50% confidence interval)	1%	2%	0%	6%
Uncertainty Aversion (80% confidence interval)	1%	2%	0%	6%
Uncertainty Aversion (95% confidence interval)	1%	1%	0%	5%
Panel C: UK				
Mean-Variance	8%	16%	0%	64%
Uncertainty Aversion (50% confidence interval)	0%	2%	0%	11%
Uncertainty Aversion (80% confidence interval)	0%	1%	0%	6%
Uncertainty Aversion (95% confidence interval)	0%	1%	0%	5%

Note: σ indicates standard deviation.

Table 10 shows the out-of-sample performance in anchored windows. For France, the pattern of out-of-sample performance is similar to that found in the broad European analysis. Both average return and standard deviation are lowered after allowing for uncertainty aversion. However, the return/risk ratios are higher under the uncertainty aversion approach. For Germany, the return/risk ratio is poor under the mean-variance approach at just 0.0548. However, under the uncertainty aversion approach, average returns increase, and the standard deviation decreases dramatically. For the UK, the return/risk ratio is poor under the mean-variance approach at just 0.0817. However, under the uncertainty aversion approach, average returns increase, and the standard deviation decreases.

For all three countries, the return/risk ratio increased when we considered uncertainty aversion. One implication of our results under uncertainty aversion is the following: Given that the estimated return and standard deviation are uncertain, even with the excellent performance of listed real estate, it is not the best option to allocate too much to the sector. The more stable allocation under uncertainty aversion yields a better return/risk ratio. In markets with periods of poor performance and high uncertainty associated with listed real estate, it is perhaps better to avoid the sector and focus on better investment opportunities.

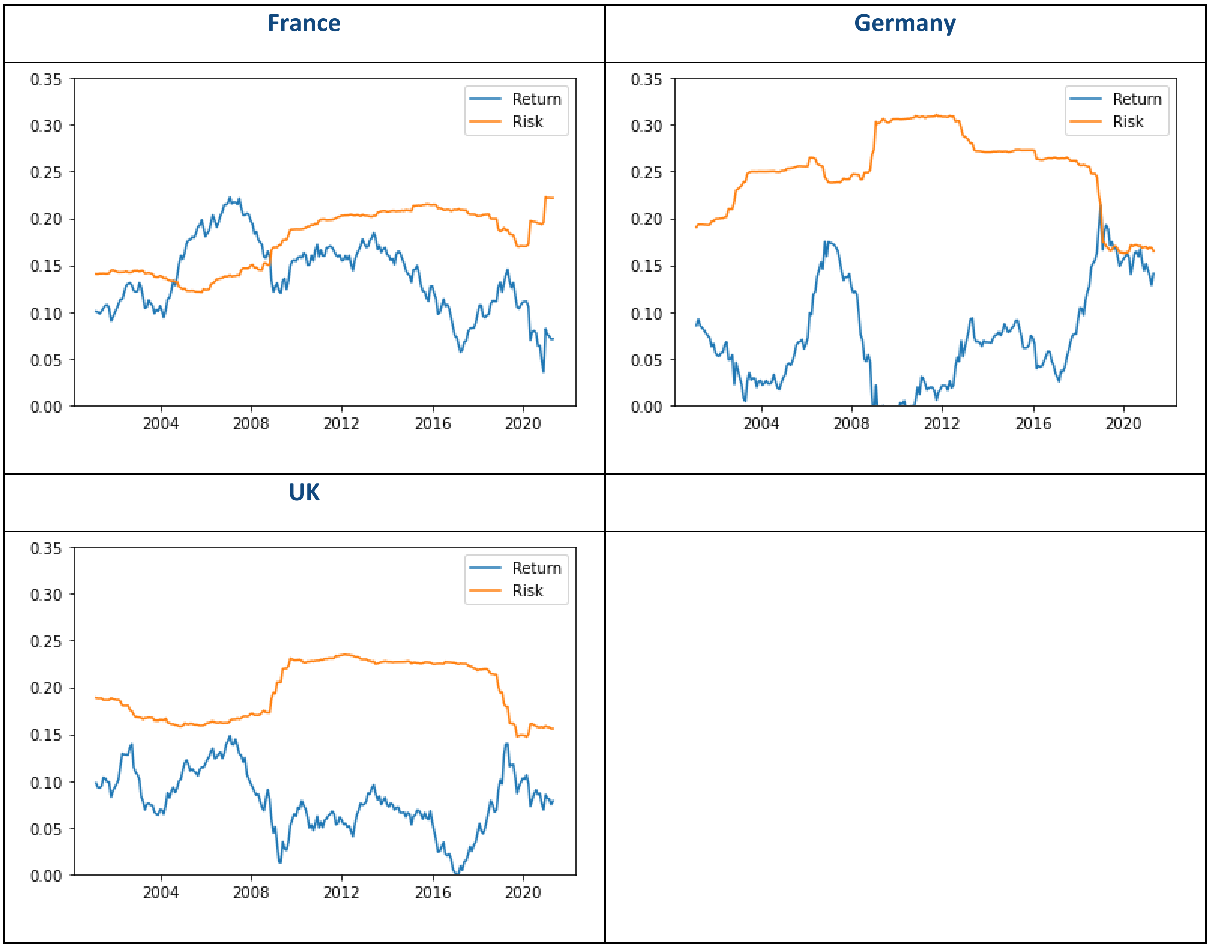
Table 10. Out-of-sample performance, anchored windows

	Mean	**σ**	$\frac{\textbf{Mean}}{\boldsymbol{\sigma}}$
Panel A: France			
Mean-Variance	0.56%	5.50%	0.1012
Uncertainty Aversion (50% confidence interval)	0.35%	2.32%	0.1524
Uncertainty Aversion (80% confidence interval)	0.37%	1.72%	0.2178
Uncertainty Aversion (95% confidence interval)	0.38%	1.47%	0.2574
Panel B: Germany			
Mean-Variance	0.18%	3.34%	0.0548
Uncertainty Aversion (50% confidence interval)	0.29%	1.06%	0.2774
Uncertainty Aversion (80% confidence interval)	0.31%	1.01%	0.3047
Uncertainty Aversion (95% confidence interval)	0.31%	1.00%	0.3122
Panel C: UK			
Mean-Variance	0.19%	2.31%	0.0817
Uncertainty Aversion (50% confidence interval)	0.38%	1.56%	0.2409
Uncertainty Aversion (80% confidence interval)	0.38%	1.55%	0.2477
Uncertainty Aversion (95% confidence interval)	0.39%	1.54%	0.2547

5.2. ROLLING WINDOW ANALYSIS

Figure 9 shows the historical annualised average returns and standard deviations of listed real estate returns using rolling windows. Before 2017, France's listed real estate had higher average returns and lower risks compared to Germany's and UK listed real estate in most of the time periods. However, there were sharp increases in average returns for both Germany's and UK listed real estate after 2017. By 2019, Germany had the highest average returns in listed real estate. Noticeably, there were sharp increases in average returns of listed real estate in all three countries during 2004-2007. The average return over the previous 10 years reached more than 20% in 2007 for France.

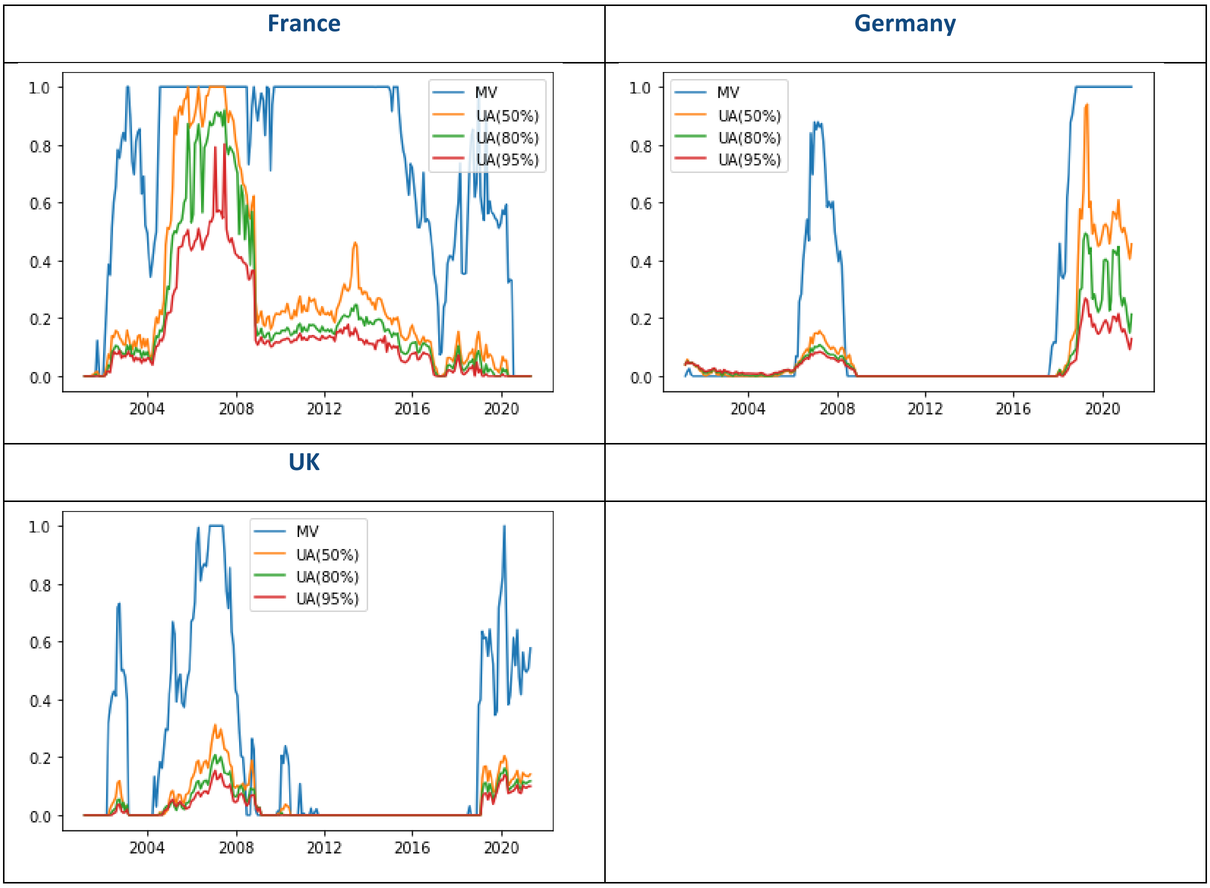
Figure 9. 120-month rolling windows annualised average returns and standard deviations of listed real estate



Source: Author's calculation

Figure 10 shows the optimal allocations to listed real estate using rolling windows with the summary statistics reported in Table 11. For France, the average optimal allocations to listed real estate are significantly lower when allowing for uncertainty aversion, at levels more consistent with institutional real estate holdings. However, the standard deviations of allocations are still high under uncertainty aversion. Real estate allocations are volatile in the 2004-2009 sub-period. This is due to the sharp increase and decline in average returns over 2004-2009 and very strong performance before 2007. Focussing on Germany and the UK, the average optimal allocations to listed real estate are higher than in the case of anchored windows. Once the large declines in the listed real estate sector's returns during the Global Financial Crisis disappeared in the 120-month rolling windows and with strong market performance, optimal allocations increased dramatically after 2018. There was a peak in allocation to UK listed real estate in 2007, this is due to the superior risk-adjusted performance of UK listed real estate before 2007. An investor following classic mean-variance allocation would have been over-invested in real estate going into the Global Financial Crisis (2007-2009) and may have divested their real estate holdings in advance of the recovery in listed real estate prices. However, the allocations of the uncertainty averse investor would have been more stable and less prone to market timing error.

Figure 10. Optimal allocations to listed real estate, rolling windows



Source: Author's calculation

Table 11. Summary statistics of optimal allocations to European listed real estate, rolling windows

	Mean	σ	Min	Max
Panel A: France				
Mean-Variance	75%	33%	0%	100%
Uncertainty Aversion (50% confidence interval)	28%	30%	0%	100%
Uncertainty Aversion (80% confidence interval)	21%	25%	0%	92%
Uncertainty Aversion (95% confidence interval)	15%	17%	0%	80%
Panel B: Germany				
Mean-Variance	21%	37%	0%	100%
Uncertainty Aversion (50% confidence interval)	8%	18%	0%	94%
Uncertainty Aversion (80% confidence interval)	5%	11%	0%	49%
Uncertainty Aversion (95% confidence interval)	3%	6%	0%	27%
Panel C: UK				
Mean-Variance	22%	31%	0%	100%
Uncertainty Aversion (50% confidence interval)	5%	7%	0%	31%
Uncertainty Aversion (80% confidence interval)	3%	5%	0%	21%
Uncertainty Aversion (95% confidence interval)	2%	4%	0%	15%

Note: σ indicates standard deviation.

Table 12 shows the out-of-sample performance using rolling windows. The patterns are similar to estimations using anchored windows.

Table 12. Out-of-sample performance, rolling windows

	Mean	σ	$\frac{\text{Mean}}{\sigma}$
Panel A: France			
Mean-Variance	0.77%	5.33%	0.1449
Uncertainty Aversion (50% confidence interval)	0.46%	3.01%	0.1516
Uncertainty Aversion (80% confidence interval)	0.42%	2.58%	0.1621
Uncertainty Aversion (95% confidence interval)	0.42%	1.95%	0.2151
Panel B: Germany			
Mean-Variance	0.12%	3.63%	0.0342
Uncertainty Aversion (50% confidence interval)	0.30%	1.56%	0.1921
Uncertainty Aversion (80% confidence interval)	0.32%	1.25%	0.2543
Uncertainty Aversion (95% confidence interval)	0.33%	1.13%	0.2884
Panel C: UK			
Mean-Variance	0.07%	3.43%	0.0208
Uncertainty Aversion (50% confidence interval)	0.29%	1.81%	0.1595
Uncertainty Aversion (80% confidence interval)	0.34%	1.63%	0.2104
Uncertainty Aversion (95% confidence interval)	0.36%	1.58%	0.2301

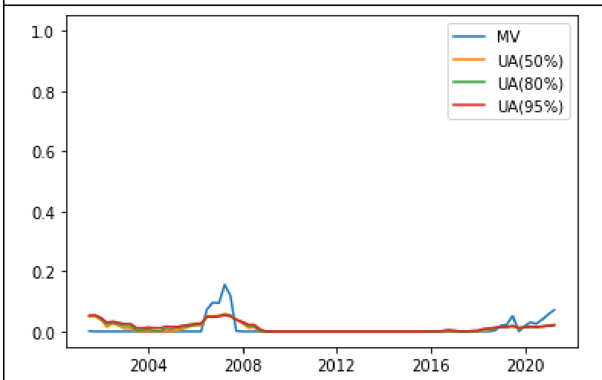
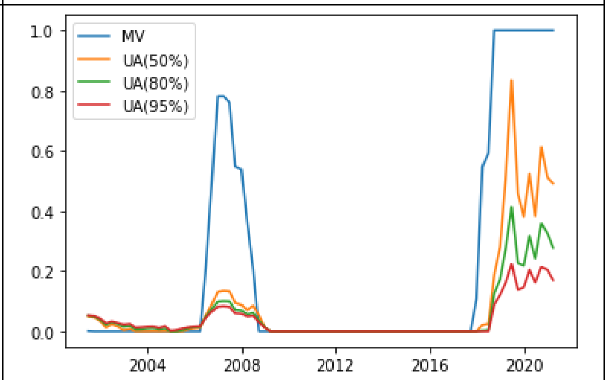
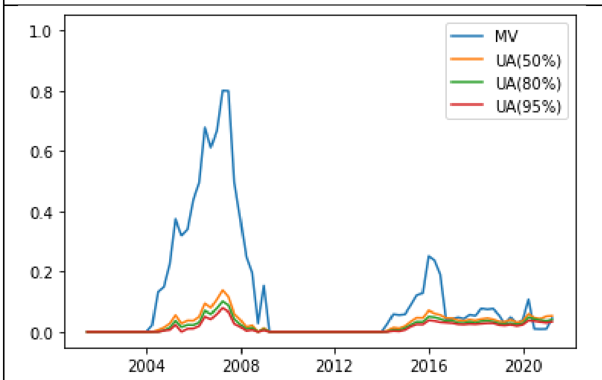
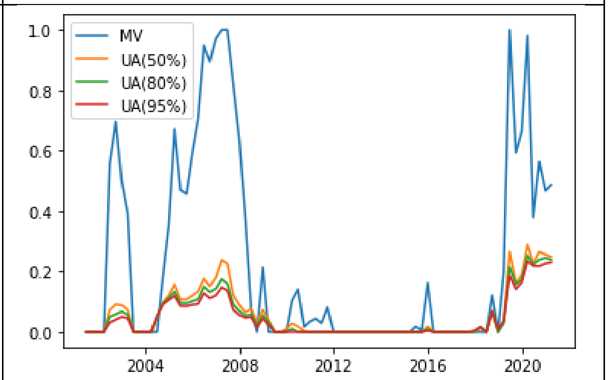
Note: σ indicates standard deviation.

5.3. EFFECT OF DATA FREQUENCY

Monthly data are used in this study. However, not all asset classes can provide reliable monthly returns. In this subsection, we lower the data frequency (using quarterly total returns) and analyse the effect on the results. In the anchored window approach, we start from a 40-quarter window, add one quarter each step, and use this historical information to estimate optimal asset allocations. We update the estimated optimal asset allocations each quarter for every additional quarter of information available. In the rolling window approach, the estimated optimal asset allocations are always based on the historical information of the last 40 quarters.

Figure 11 shows the optimal allocations to listed real estate using anchored windows. For France, the peak optimal allocation in 2007 under uncertainty aversion is significantly lower than when using monthly data. The optimal allocations under uncertainty aversion during other periods are slightly lower. Using quarterly data produces more stable allocations than using monthly data. For both Germany and UK, there are almost no difference compared to the monthly data frequency. Figure 12 shows the optimal allocations to listed real estate using rolling windows. The patterns are similar to those with anchored windows. For brevity, the summary statistics of optimal allocations and out-of-sample performances are not reported.

The results imply that sample size plays an important role in the uncertainty aversion framework. As the sample size decreases, the confidence interval gets larger for any given confidence level, while the estimation of the mean return becomes less precise and less reliable. In the case of France, superior returns are accompanied by large fluctuations in returns (especially in 2007); using fewer observations (quarterly data) reduces the precision of estimation of the mean return and reduces the optimal allocation of listed real estate. It is for this reason that there is a significant drop in the peak allocations in 2007 when compared to those estimated using monthly data.

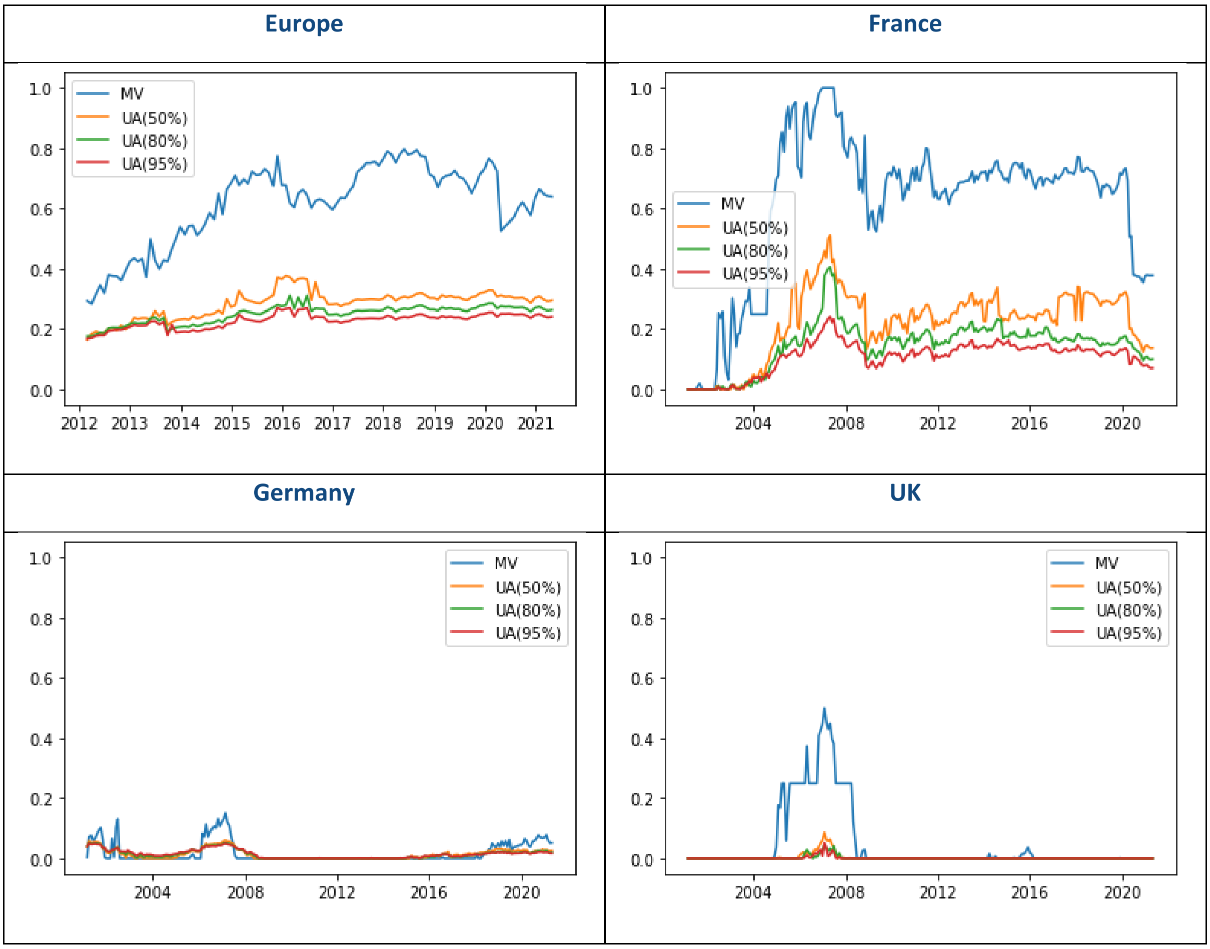
Figure 11. Optimal allocations to listed real estate with quarterly data, anchored windows	**Figure 12.** Optimal allocations to listed real estate with quarterly data, rolling windows
France	**France**
	
Germany	**Germany**
	
UK	**UK**
	

Source: Author's calculation

5.4. EFFECT OF RISK AVERSION

We choose the risk aversion parameter γ to be 2 in this study. In this subsection, we increase risk aversion (setting γ to be 3) and analyse the effect on the results. Figure 13 shows the optimal allocations to listed real estate using anchored windows. For the allocations to European listed real estate, the gaps in allocations between the mean-variance approach and the uncertainty aversion approach are smaller than when we use $\gamma = 2$. This is mainly due to the lower allocation to listed real estate under the classic mean-variance approach. For a more risk averse investor, he/she shifts his/her asset allocations from riskier assets to safer assets. For France, we find the same result, the gaps in allocations between the mean-variance approach and the uncertainty aversion approach are smaller when $\gamma = 3$ than when $\gamma = 2$. The fluctuation in allocations with a confidence interval of 50% is significantly reduced compared to the case of $\gamma = 2$. The fluctuation in allocations with a confidence interval of 95% is similar to the case of $\gamma = 2$. For both Germany and the UK, there are almost no differences in allocations compared to the case of $\gamma = 2$, the allocations to listed real estate remain very low.

Figure 13. Optimal allocations to listed real estate with a risk aversion parameter 3, anchored windows

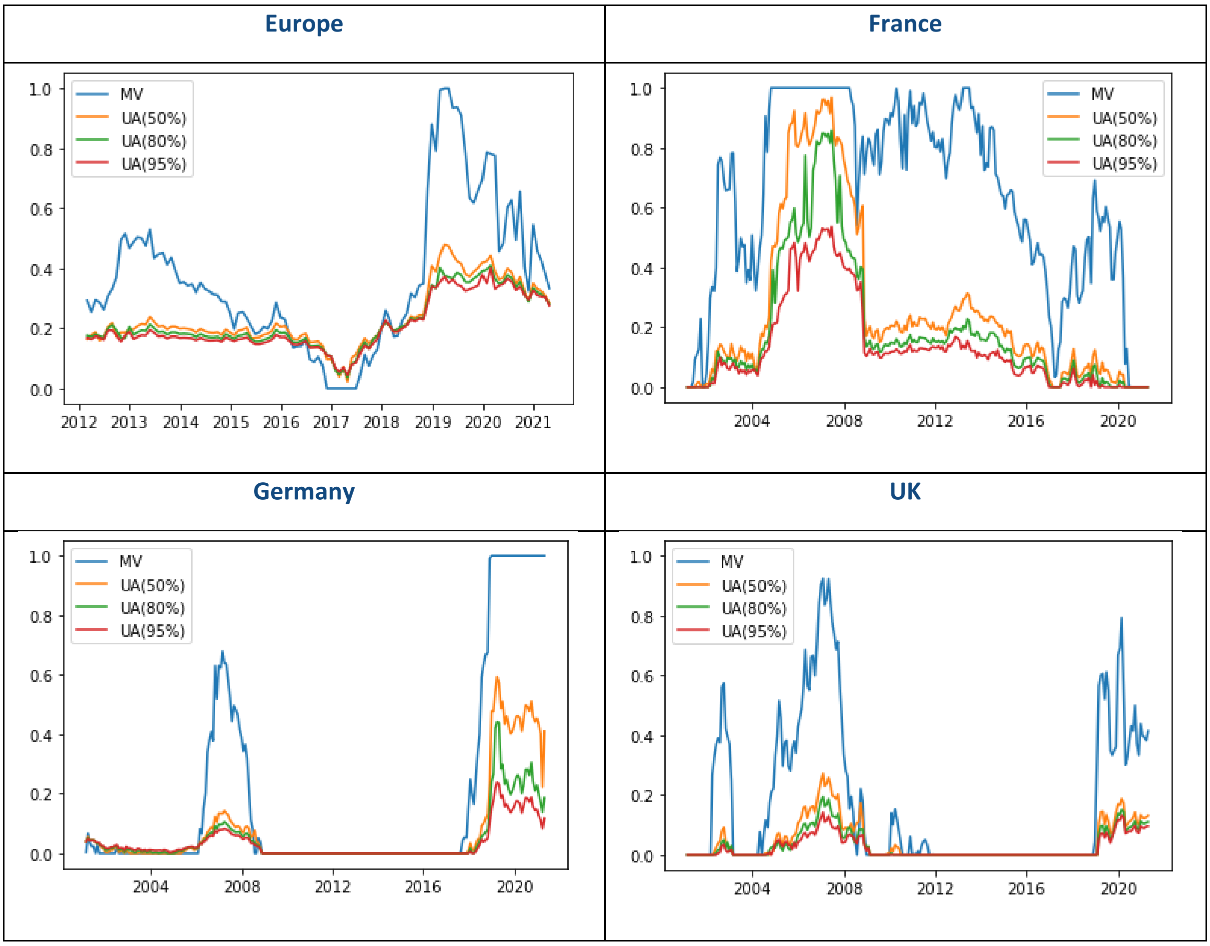


Source: Author's calculation

Figure 14 shows the optimal allocations to listed real estate using rolling windows. For the allocations to European listed real estate, it is similar to the case of anchored windows, the gaps in allocations between the mean-variance approach and the uncertainty aversion approach are smaller than when we use $\gamma = 2$. For France, one noticeable result is that the fluctuation in allocations with a confidence interval of 95% is significantly lower than with the $\gamma = 2$ case. Focusing on Germany, the fluctuation in allocations with a confidence interval of 50% is significantly lower than with $\gamma = 2$ but there is little difference in allocations with a confidence interval of 95% between $\gamma = 2$ and $\gamma = 3$. For the UK, there is almost no difference in allocations compared to the case of $\gamma = 2$. For brevity, the summary statistics of optimal allocations and out-of-sample performances are not reported.

The implication of the results is the following: If the uncertainty aversion approach already yields stable allocations, increasing risk aversion should not affect the allocations. However, if the uncertainty aversion approach does not yield stable allocations, increasing risk aversion increases that stability. In order to reach more stable allocations, risk aversion and uncertainty aversion might seem to act as substitutes. However, they represent different concepts and, thus, may not always act as substitutes. For example, Berger and Eeckhoudt (2021) discussed how risk aversion and uncertainty aversion affect the value of diversification differently.

Figure 14. Optimal allocations to listed real estate with a risk aversion parameter 3, rolling windows



Source: Author's calculation

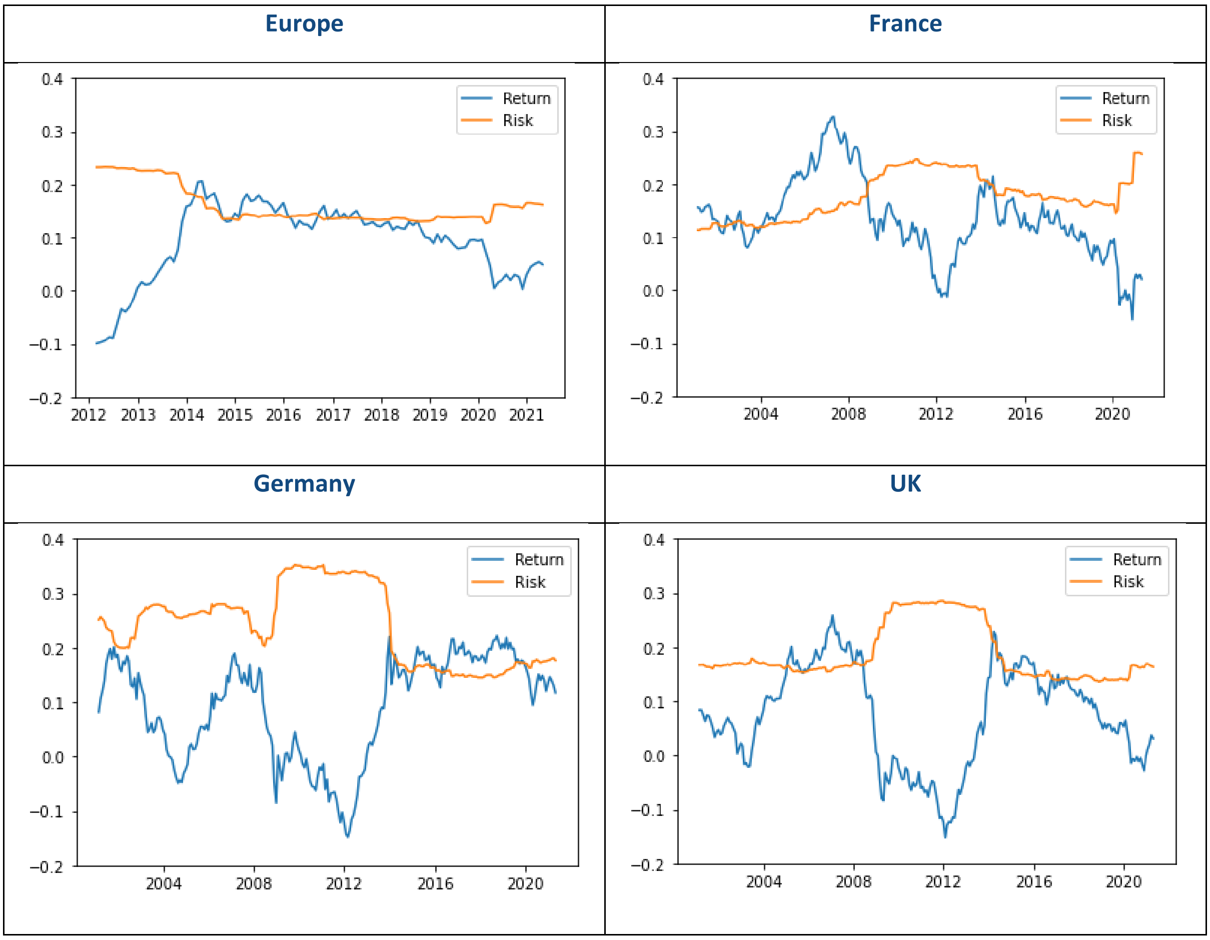
5.5. EFFECT OF SHORTER ROLLING WINDOW LENGTH

As we discussed before, using a longer window for estimating can ignore structural changes and put less weight on recent market movements. Thus, in this subsection, we use 60-month (5 years) rolling windows instead of 120-month rolling windows. One potential benefit of using a shorter window is that the optimal allocations could pick up assets with relatively good recent risk-adjusted performances[4]. Historic annualised average returns and standard deviations are presented in Figure 15. For the European listed real estate, the 60-month average returns increased sharply before 2014 and were in a modest downtrend after 2014. The pattern of 60-month average returns for France and UK are very similar. However, UK listed real estate is riskier than France's listed

[4] Of course, it also increases the risk of being over- or under-invested at a cyclical market turning point. This rests on the extent of momentum and return persistence in each asset market.

real estate before 2020. For Germany, the 60-month average returns had large fluctuations before 2015 and stabilised after that.

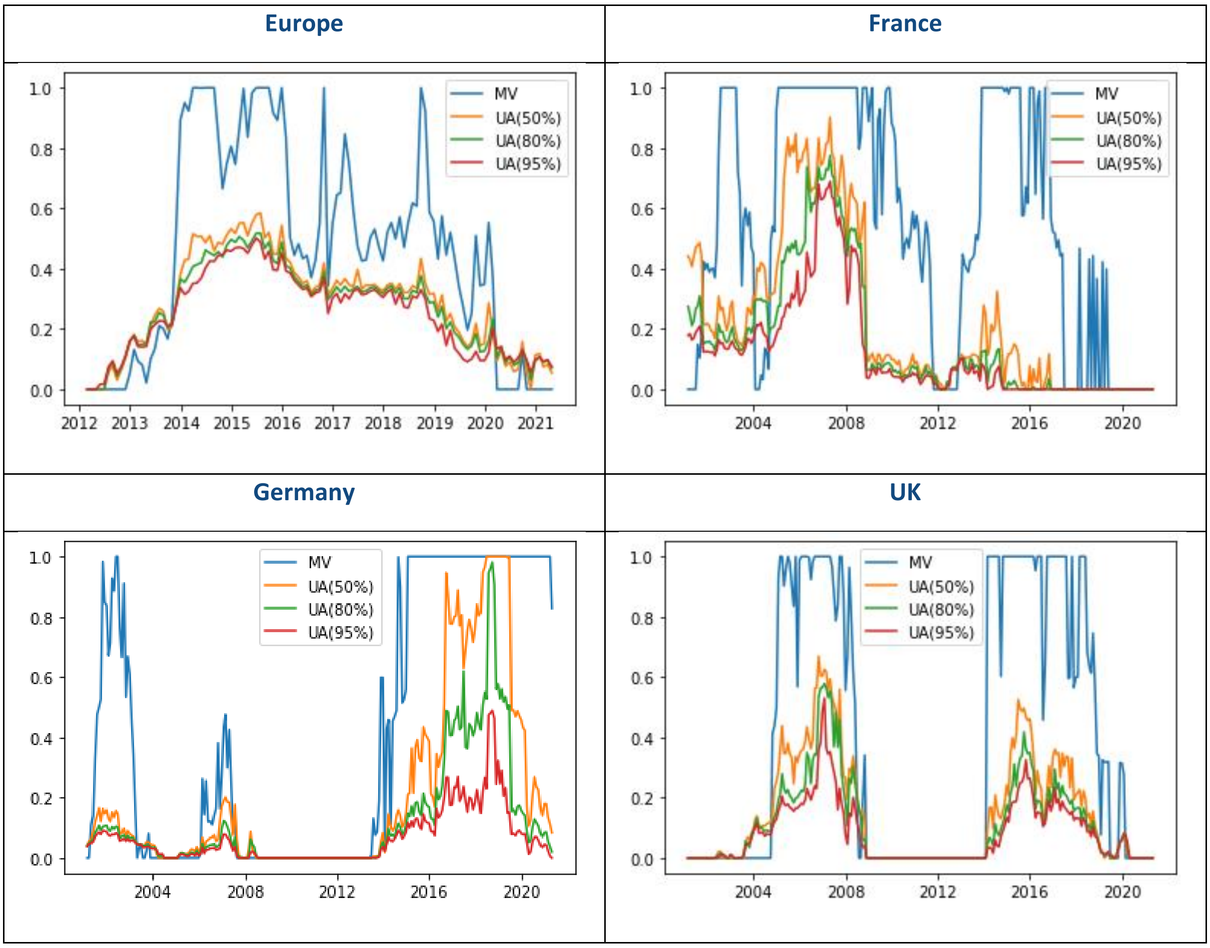
Figure 15. 60-month rolling windows annualised average returns and standard deviations of listed real estate



Source: Author's calculation

Figure 16 shows the optimal allocations to listed real estate using 60-month rolling windows. Focussing on the broad European market, the optimal allocations of European listed real estate are still stable over time. However, the pattern is somewhat different from the estimation using 120-rolling windows. In the case of 60-month rolling windows, allocation to listed real estate reached a peak in 2015 and gradually decreases after 2015. In the case of 120-month rolling windows, allocation to listed real estate reached the peak in 2019. The reason that 60-month rolling windows yield a peak in allocation to listed real estate earlier is that the impact of the Global Financial Crisis (2007-2009) period drops out earlier with a shorter estimation window. Focussing on France, the allocations pattern is similar to the case of 120-month rolling window. However, the optimal allocation to listed real estate is lower with 60-month rolling windows compared to that with 120-month rolling windows after 2009. Focusing on Germany and the UK, the allocations pattern is similar to the case of 120-month rolling window. However, the allocations to the listed real estate sector start to above 0 in 2014 which is earlier than the case of 120-month rolling window. Again, this is because the Global Financial Crisis (2007-2009) period dropped out earlier with a shorter estimation window. With a shorter estimation window, the allocations of mean-variance approach become even more volatile. However, the allocations of uncertainty aversion approach are still relatively stable, especially in the case of allocations with the 95% confidence interval. For brevity, the summary statistics of optimal allocations and out-of-sample performances are not reported.

Figure 16. Optimal allocations to listed real estate, 60-month rolling windows



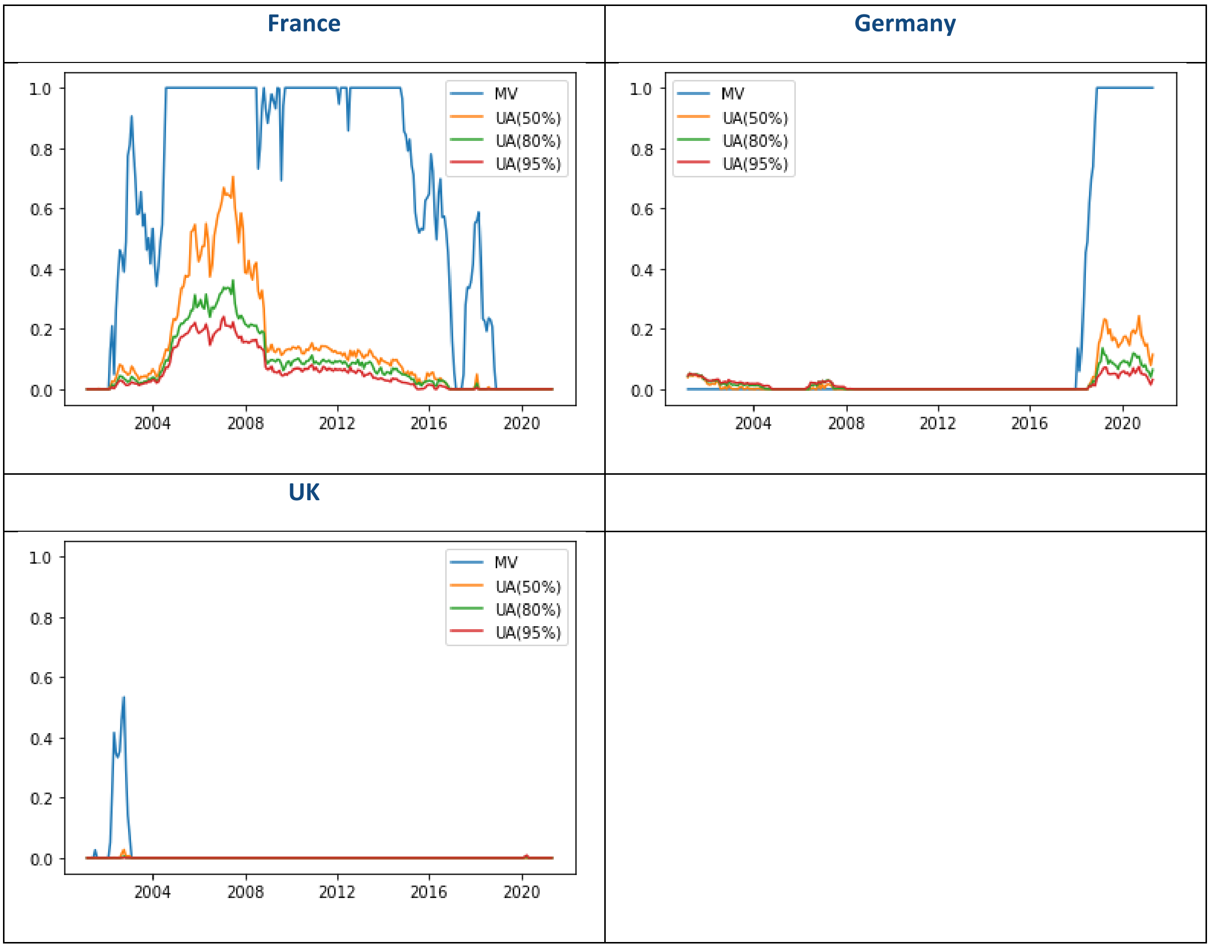
Source: Author's calculation

5.6. ASSETS FROM MULTIPLE COUNTRIES

From the previous analysis, two facts motivate the analysis in this section. First, the use of rolling windows allows us to capture recent market movements and allocate to those asset classes that have performed better in recent times. Second, there was a decline in the listed real estate sector performance in France after 2018 which yielded 0% allocations to listed real estate. However, by contrast, there was a sharp increase in performance in the listed real estate sector in Germany and UK after 2017 yielding a higher allocation to listed real estate under uncertainty aversion compared to the near zero allocations in the previous period. Thus, in this section, we combine all the assets in France, Germany and UK and form portfolios based on those assets using rolling windows.

Figure 17 shows the optimal allocations to France's listed real estate, Germany's listed real estate and UK listed real estate using 120-month rolling windows. The optimal allocations with a confidence interval of 95% are relatively stable. Although the allocation to France's listed real estate reached to 0 % in 2017, allocation to Germany's listed real estate picked up after 2018. The results show a modest shift in listed real estate allocation from France to Germany after 2016. The UK listed real estate sector did not play any role in the optimal portfolios. The analysis has several implications. First, under uncertainty aversion with a large enough level of uncertainty, we can form a portfolio based on a large number of assets, it will give us relatively stable allocations and less extreme allocations, even with an asset that has superior performance. Second, following from the first point, we can add listed real estate of many countries or different sectors of listed real estate in the portfolio optimisation process. Under uncertainty aversion, the model will give us a modest shift in allocations from one country to another or from one sector to another. This allows us to dynamically allocate funds into an asset that has a better performance without being too extreme in allocations and incurring potentially substantial rebalancing costs.

Figure 17. Optimal allocations to listed real estate with quarterly data, rolling windows



Source: Author's calculation

6. CONCLUSION

The classic mean-variance approach for portfolio optimisation often suggests extreme asset allocations that fluctuate significantly over time. By adopting the uncertainty aversion framework, optimal allocations to listed real estate show much greater stability. This is particularly useful when using rolling windows to provide parameters for the asset allocation exercise, since these generate greater variation in returns and less stable inputs for optimisation. The uncertainty aversion approach can better deal with fluctuating inputs and, hence, produces more stable asset allocations. An important implication of more stable asset allocations over time is fewer portfolio rebalancing events, incurring lower transaction costs. In addition, we show that the out-of-sample return/risk ratios are higher under the uncertainty aversion approach compared to the mean-variance approach.

Given the magnitude of fluctuations in returns of the listed real estate sector in Europe, the estimated returns and volatilities can be very different across different periods. In other words, those point estimates are associated with uncertainty. By considering uncertainty of estimation using historical information and being averse to uncertainty, we show that, even with excellent past performance in listed real estate, it should be cautious to allocate too much to the sector. A transition from a listed real estate boom period to a decline period would significantly reduce the allocation to listed real estate if uncertainty aversion is not considered. Thus, the resultant uncertainty aversion portfolios are more stable over time and deliver a higher return/risk ratio. Given the heterogeneity in the performance of listed real estate among different European countries over different periods, an investor could form a portfolio including listed real estate from many countries. Since the uncertainty aversion approach gives us relatively stable allocations over time (even with many assets), the model will give us more modest shifts in allocations in listed real estate from one country to another over time. For example, the results of the multiple country analysis (UK, France and Germany) show a modest shift in listed real estate allocation from France to Germany after 2016. Thus, the uncertainty aversion model allows dynamic allocation of funds into the listed real estate sector with better performance without being too extreme in allocations and avoiding significant rebalancing costs.

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Appendix

APPENDIX A: THEORETICAL FRAMEWORKS OF PORTFOLIO OPTIMISATIONS

According to the classical mean-variance model, the optimal portfolio for N risky assets, w, is given by the solution of the following optimisation problem:

$$\max_{w} w^T\mu - \frac{\gamma}{2} w^T \Sigma w$$

$$\text{Subject to: } w^T 1_N = 1$$

where μ is the N-vector of the expected returns, Σ is the $N \times N$ covariance matrix, and the scalar γ is the risk aversion parameter.

Under uncertainty aversion, the optimisation problem becomes:

$$\max_{w} \min_{\mu} w^T\mu - \frac{\gamma}{2} w^T \Sigma w$$

$$\text{Subject to: } f(\mu, \hat{\mu}, \Sigma) \leq \epsilon$$

$$w^T 1_N = 1$$

where f(.) is a vector-valued function that characterises the constraint, μ is the true expected return, $\hat{\mu}$ is the estimated expected return, and ϵ is a vector of constraints that reflects both the investor's uncertainty and his aversion to uncertainty. The parameter ϵ should be understood as the product of uncertainty aversion (common across assets) and uncertainty (asset-specific). The uncertainty aversion is not observable and we normalise the degree of uncertainty aversion to 1. In the model, the investor does not choose the degree of uncertainty aversion but only the asset-specific level of uncertainty. In this case, Garlappi et al. (2007) claim that the parameter ϵ can be interpreted as the size of a confidence interval.

The fact that the optimisation problem is maxmin under uncertainty aversion is explained by Gilboa and Schmeidler (1989). Being uncertainty averse, she/he takes into account the minimal expected utility (over all priors in the set) while evaluating a bet. In our case, for any given weight, the investor evaluates the investment opportunity by choosing expected return μ that gives the lowest portfolio returns. In other words, we find out the worst scenario based on any given weight first. Then amongst those worst scenarios, we choose the weight that gives us the highest portfolio returns.

In order to implement the above uncertainty aversion maximisation problem in practice, Garlappi et al. (2007) proposed two methods: (1) state the confidence interval for the expected returns of assets individually; (2) state the confidence interval for the expected return of assets jointly. In this study, we adopt method (2) and the constraint $f(\mu, \hat{\mu}, \Sigma) \leq \epsilon$ can be expressed as:

$$\frac{T(T-N)}{(T-1)N} (\hat{\mu} - \mu)^T \Sigma^{-1} (\hat{\mu} - \mu) \leq \epsilon$$

In other words, the above constraint corresponds to the probabilistic statement

$$P\left[\frac{T(T-N)}{(T-1)N} (\hat{\mu} - \mu)^T \Sigma^{-1} (\hat{\mu} - \mu) \leq \epsilon\right] = 1 - p$$

where $1 - p$ can be interpreted as confidence level. If Σ is estimated by the sample variance-covariance matrix, then $\frac{T(T-N)}{(T-1)N} (\hat{\mu} - \mu)^T \hat{\Sigma}^{-1} (\hat{\mu} - \mu)$ follows an F-distribution with N and T-N degrees of Freedom ($F_{N,T-N}$), where N is the number of assets and T is the number of observations for each asset. Choosing a confidence interval is the same as choosing ϵ. For example, when T=120 and N=4, ϵ=2.45 corresponds to a confidence interval of 95%.

Following method (2), the maximation problem can be simplified as the following (see proof in Garlappi et al., 2007):

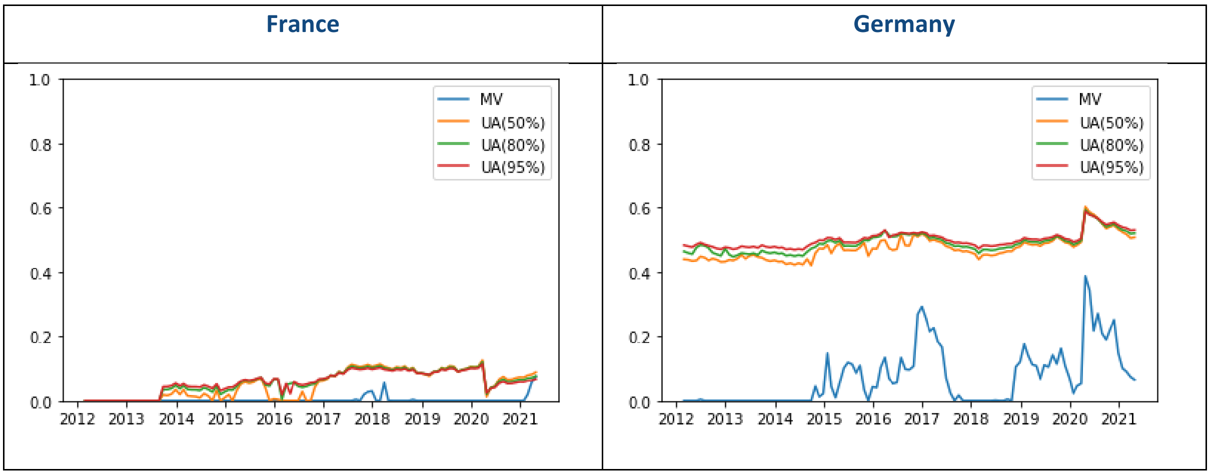
$$\max_{w} w^T\mu - \frac{\gamma}{2} w^T \Sigma w - \sqrt{\varepsilon w^T \Sigma w}$$

$$\text{Subject to: } w^T 1_N = 1$$

where $\varepsilon = \epsilon \frac{(T-1)N}{T(T-N)}$

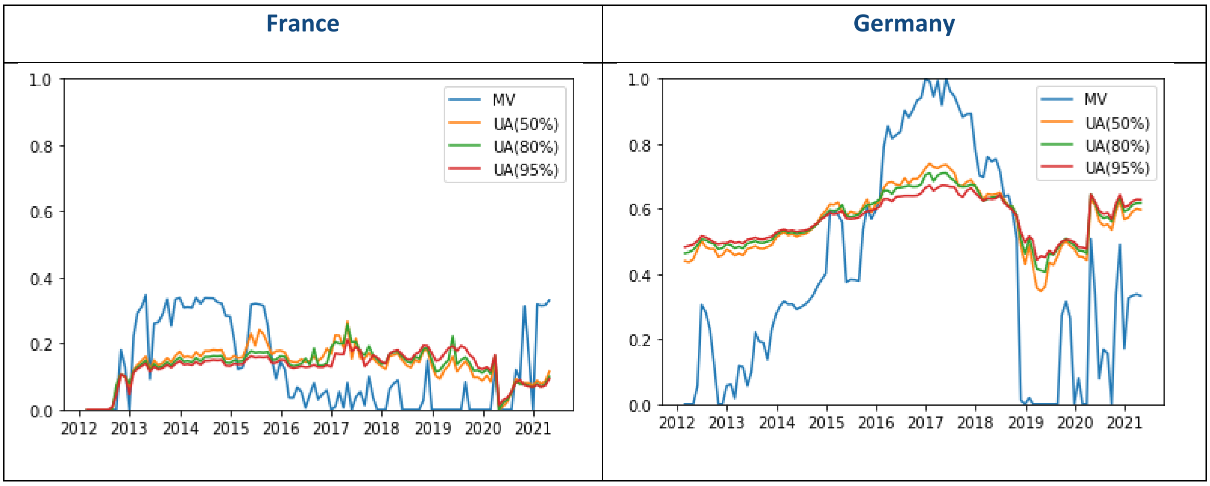
APPENDIX B: OPTIMAL ALLOCATIONS TO STOCKS AND GOVERNMENT BONDS

Figure B1. Optimal allocations to stocks and government bonds, anchored windows



Source: Author's calculation

Figure B2. Optimal allocations to stocks and government bonds, rolling windows



Source: Author's calculation